Talk 18th January: Rational filling conditions for torsion free sheaves

recall: Let R be a complete discrete valuation ring over S and let ϖ be a uniformizer, we call

$$Y_{\Theta_R} := \operatorname{Spec}(R[t]) \quad \text{and} \quad Y_{\overline{ST}_R} := \operatorname{Spec}(R[s,t]/st-\varpi)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\Theta_R := [Y_{\Theta_R}/\mathbb{G}_m] \quad \text{and} \quad \overline{ST}_R := [Y_{\overline{ST}_R}/\mathbb{G}_m]$$

definition: Let $\mathcal{M}$ be either $\mathcal{C}oh$, $\mathcal{P}air$ or $\Lambda\mathcal{C}oh$. Let $\mathfrak{X}$ be either Θ_R or $\overline{ST}_R$. We say that $\mathcal{M}$ admits the $\mathfrak{X}$ rational filling condition if:

For all morphisms $f: \mathfrak{X}\setminus(0,0) \to \mathcal{M}$ there exists a morphism $g: \mathfrak{X} \to \mathcal{M}_{rat}$ and a 2-commutative diagram of pseudofunctors:

$$\begin{tikzcd}
\mathfrak{X}\setminus(0,0) \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{M} \arrow[d] \\
\mathfrak{X} \arrow[r, dashed, "g"'] & \mathcal{M}_{rat}
\end{tikzcd}$$

where j is the inclusion $\mathfrak{X}\setminus(0,0) \hookrightarrow \mathfrak{X}$

Lemma 4.6: Suppose that the stack $\mathcal{C}oh^d(X)$ admits Θ_R rational filling condition (resp. $\overline{ST}_R$) then both $\mathcal{P}air^d_{\mathcal{A}}(X)$ and $\Lambda\mathcal{C}oh^d(X)$ admit Θ_R rational filling condition (resp. $\overline{ST}_R$).

proof idea: Note that a $\mathbb{G}_m$-equivariant morphism $\mathcal{Y} \setminus (0,0) = W \longrightarrow \mathcal{M}$ is equivalent to a W-pure sheaf $\mathcal{F}$ of dim d on X_W and a $\mathbb{G}_m$-equivariant morphism that gives $\mathcal{F}$ the structure that is lacking to belong to $\mathcal{M}$. By the hypothesis we just have to extend β to the rational "image", achieved by [Sto20, Tag 0E9I]

Now we study the filling condition in the case when the fibers of $X \to S$ are geometrically integral of dimension d. In this case pure sheaves are just torsion free.

Lemma. Suppose that the morphism $X \to S$ is flat with geometrically integral fibers of dimension d. Then $\mathcal{C}oh^d(X)$ admits both filling conditions.

proof: Let R be a complete discrete valuation ring over S. Let k be the residue field of R and let K be its fraction field. Let $\mathcal{X}$ be either Θ_R or ST_R and $\mathcal{Y} = \mathcal{Y}_{\mathcal{X}}$. We write W for the open complement of $(0,0)$ in $\mathcal{Y}$ and $j: W \hookrightarrow \mathcal{Y}$. Let X_R be the resulting pullback through $\operatorname{spec} R \to S$, $X_{\mathcal{Y}}$ is obtained by further pullback through $\mathcal{Y} \to \operatorname{spec} R$.

Suppose that we are given $f: \mathcal{X} \to \mathcal{C}oh^d(X)$ this ammounts to a $\mathbb{G}_m$-equivariant W-flat family $\mathcal{F}$ of torsion free sheaves on X_W.

Note also that a principal subscheme of $X_{\mathcal{Y}}$ is regular with respect to a torsion free sheave iff it is a $\mathcal{Y}$-relative Cartier divisor.

Thus proving the filling condition is equivalent to building:

(1) A $\mathbb{G}_m$-equivariant $\mathcal{Y}$-pure sheaf $\mathcal{E}$ of dimension d on $X_{\mathcal{Y}}$.

(2) A $\mathbb{G}_m$-equivariant relative Cartier Divisor $D \hookrightarrow X_{\mathcal{Y}}$.

(3) A $\mathbb{G}_m$-equivariant morphism $\psi: \mathcal{E}|_{X_W} \longrightarrow \mathcal{F}$ st. $\psi|_{X_W \setminus D_W}$ is an iso.

The idea: for (1), note that since $X_{(0,0)}$ is integral then for most points in $X_{(0,0)}$ $\widetilde{\mathcal{F}} := (j_x)_* \mathcal{F}$ is a free graded module (i.e. it is of the form $\bigoplus_{i\in I} \mathcal{O}\langle m_i\rangle$)

So we extend this to the whole X_Y by setting

$$\mathcal{E} = \bigoplus_{i\in I} \mathcal{O}_{X_Y}\langle m_i\rangle.$$

for (2), note that we can find a big enough $n >> 0$ st. there is a section $s_0 \in H^0(X_{(0,0)}, \mathcal{O}(n))$ which vanishes at points where $\widehat{\mathcal{F}}$ is not the free graded module described before (inside $X_{(0,0)}$) this section is extended to X_Y "by $\mathbb{G}_m$-invariance" and thus its vanishing locus is a Cartier divisor

for (3), by construction they are both isomorphic outside of D, but inside of $X_{(0,0)}$, call Ψ this iso.
So we extend Ψ in two steps, first extend to $X_Y \setminus D$ which can be done because the open complement of $X_Y \setminus D$ is affine and the fact that $\mathcal{E}$ restricted to this open set is locally free. Then extend to the whole X_Y by twisting $\mathcal{E}$ if necessary, this extension will be called $\bar{\Psi}$.
Now we adjust the construction by removing the points where $\bar{\Psi}$ is not surjective

↳ This turns out to be a Cartier divisor, D'

adjust the divisor by setting $D+D'$ as the divisor we are looking for. Then by construction $\bar{\Psi}$ is an iso away from $D+D'$ between $\mathcal{E}$ and $\mathcal{F}$.

A different set of numerical invariant

- How to construct polynomial numerical invariants on a stack from a sequence of rational line bundles $(\mathcal{L}_n)_{n\in\mathbb{Z}} \in \mathrm{Pic}(\mathcal{M})\otimes\mathbb{Q}$.

$\to$ for each $g : (B\mathbb{G}_m^q)_k \longrightarrow \mathcal{M}$ the pullback $g^*\mathcal{L}_n$ is determined by a character in $X^*(\mathbb{G}_m^q)\otimes\mathbb{Q}\cong\mathbb{Q}^q$

i.e a q-tuple $(w_n^i)_{i=1}^q$ of rat numbers (called the weight of $g^*\mathcal{L}_n$).

If the $\mathcal{L}_n$ are well chosen st: fixing i_0 results in a $w_n^{i_0} : \mathbb{Z}\to\mathbb{Q}$ a $\mathbb{Z}$-valued polynomial then we can define an $\mathbb{R}$-linear function:

$$L_g : \mathbb{R}^q \longrightarrow \mathbb{R}[n]$$
$$(r_i)_{i=1}^q \to \sum_{i=1}^q r_i \cdot w_n^{(i)}$$

Moreover if we have a rational quadratic norm on graded points

i.e. for each $p\in\mathcal{M}(k)$ a positive definite quadratic norm $b_p(-)$ with rat coefs deffined on $\mathbb{R}^q$ with some compatibility conditions.

$\Rightarrow$ in this case: for all nondegenerate $g : (B\mathbb{G}_m^q)_k \to \mathcal{M}$ with corresponding $\gamma : (\mathbb{G}_m^q)_k \to \mathrm{Aut}(g|_{\mathrm{Spec}\,k})$ we

set :
$$\nu_\gamma(\bar{r}) = \frac{L_g(\bar{r})}{\sqrt{b_\gamma(\bar{r})}}$$

There are various naturally defined families of line bundles on $\mathcal{Coh}^d(X)$

Let $n \in \mathbb{Z}$ then the line bundle M_n on $\mathrm{Coh}^d(X)$ is given by:

Let T be an S-scheme and let

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "\pi_T"'] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

$f : T \to \mathrm{Coh}^d(X)$ correspond to a T pure sheaf $\mathcal{F}$ on X_T

Then $f^* M_n := \det R_{\pi_T *}(\mathcal{F}(n))$

$\to M_n \in \mathrm{Pic}(\mathrm{Coh}^d(X))$ is a polynomial in the variable n of degree $d+1$ with values in Pic. More precisely we can decompose it as:

$$M_n = \bigotimes_{i=0}^{d+1} b_i^{\binom{n}{i}} \quad \text{with } b_i := \bigotimes_{j=0}^{i} M_j^{(-1)^j \binom{i}{j}}$$

and let L_n be st: $\quad f^* L_n := f^* M_n \otimes f^* b_d^{-\otimes \bar{p}_{\mathcal{F}}(n)} \quad$ where $\bar{p}_{\mathcal{F}}$ is the reduced Hilbert polynomial

additionally let us define a rational quadratic norm on graded objects of $\mathrm{Coh}^d(X)$

def: Let $g : (B\mathbb{G}_m^q)_K \to \mathrm{Coh}^d(X)$ be a $\mathbb{Z}^q$ graded pure sheaf $\tilde{\mathcal{F}} = \bigoplus \tilde{\mathcal{F}}_{\bar{m}}$ of dim d on X_K.

we define $b(g)$ by:

$$b(g)v := \sum_{m \in \mathbb{Z}^q} \mathrm{rk}\, \tilde{\mathcal{F}}_{\bar{m}} \cdot (\bar{m} \cdot_{std} v)^2$$

Note that the numerical invariant obtained in this way is the natural generalization of the previously defined cohomological invariants, we just changed the ordered set where they are valued. It is worth mentioning that they will enjoy all the benefits of the cohomological invariants if the polynomial $w_m^{i_0}$ and ultimately if the family (L_n) has some boundedness condition (when $n \to \infty$). See 9.6.7. Polynomial valued numerical invariants [HL 18]